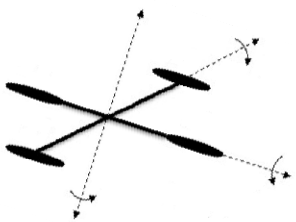
Laplace Transform -3



Translation of Laplace Transform

THEOREM 4.6 First Translation Theorem

If $\mathscr{L}\{f(t)\} = F(s)$ and a is any real number, then

$$\mathscr{L}\{e^{at}f(t)\} = F(s-a).$$

PROOF The proof is immediate, since by Definition 4.1

$$\mathscr{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a). \qquad \square$$

If we consider s a real variable, then the graph of $F(s-a)$ is the graph of $F(s)$ hifted on the s-axis by the amount $|a|$. If $a > 0$, the graph of $F(s)$ is shifted a units to ıe right, whereas if $a < 0$, the graph is shifted $|a|$ units to the left. See Figure 4.10.

For emphasis it is sometimes useful to use the symbolism

$$\mathscr{L}\{e^{at}f(t)\} = \mathscr{L}\{f(t)\}_{s \to s-a},$$

/here $s \to s-a$ means that in the Laplace transform $F(s)$ of $f(t)$ we replace the ymbol s wherever it appears by $s-a$.

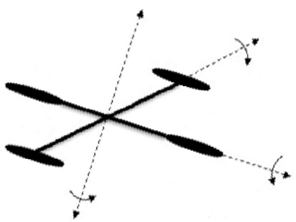
Example 1 **Using the First Translation Theorem**

Evaluate (a) $\mathscr{L}\{e^{5t}t^3\}$ (b) $\mathscr{L}\{e^{-2t}\cos 4t\}$.

SOLUTION The results follow from Theorems 4.1 and 4.6.

(a) $$\mathscr{L}\{e^{5t}t^3\} = \mathscr{L}\{t^3\}_{s \to s-5} = \left.\frac{3!}{s^4}\right|_{s \to s-5} = \frac{6}{(s-5)^4}$$

(b) $$\mathscr{L}\{e^{-2t}\cos 4t\} = \mathscr{L}\{\cos 4t\}_{s \to s-(-2)} = \left.\frac{s}{s^2+16}\right|_{s \to s+2} = \frac{s+2}{(s+2)^2+16}.$$



Example) $\mathscr{L}^{-1}\left\{\dfrac{2s+5}{(s-3)^2}\right\}$

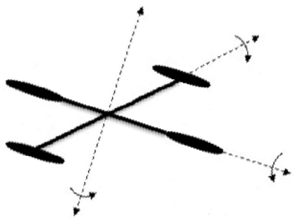
$$\frac{2s+5}{(s-3)^2} = \frac{2}{s-3} + \frac{11}{(s-3)^2} \tag{2}$$

and
$$\mathscr{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = 2\,\mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\} + 11\mathscr{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}. \tag{3}$$

Now $1/(s-3)^2$ is $F(s) = 1/s^2$ shifted 3 units to the right. Since $\mathscr{L}^{-1}\{1/s^2\} = t$, it follows from (1) that

$$\mathscr{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} = \mathscr{L}^{-1}\left\{\left.\frac{1}{s^2}\right|_{s \to s-3}\right\} = e^{-3t}t.$$

Finally, (3) is
$$\mathscr{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = 2e^{-3t} + 11e^{-3t}t. \tag{4}$$



Example 3 Initial-Value Problem

Solve $y'' - 6y' + 9y = t^2e^{3t}$, $y(0) = 2$, $y'(0) = 6$.

SOLUTION Before transforming the DE note that its right-hand side is similar to the function in part **(a)** of Example 1. We use Theorem 4.5, the initial conditions, simplify, and then solve for $Y(s) = \mathscr{L}\{f(t)\}$:

$$\mathscr{L}\{y''\} - 6\mathscr{L}\{y'\} + 9\mathscr{L}\{y\} = \mathscr{L}\{t^2e^{3t}\}$$

$$s^2Y(s) - sy(0) - y'(0) - 6[sY(s) - y(0)] + 9Y(s) = \frac{2}{(s-3)^3}$$

$$(s^2 - 6s + 9)Y(s) = 2s + 5 + \frac{2}{(s-3)^3}$$

$$(s-3)^2Y(s) = 2s + 5 + \frac{2}{(s-3)^3}$$

$$Y(s) = \frac{2s+5}{(s-3)^2} + \frac{2}{(s-3)^5}.$$

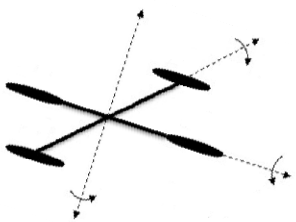
$$Y(s) = \frac{2}{s-3} + \frac{11}{(s-3)^2} + \frac{2}{(s-3)^5}.$$

Thus $$y(t) = 2\mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\} + 11\mathscr{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} + \frac{2}{4!}\mathscr{L}^{-1}\left\{\frac{4!}{(s-3)^5}\right\}.$$

From the inverse form (1) of Theorem 4.6, the last two terms are

$$\mathscr{L}^{-1}\left\{\left.\frac{1}{s^2}\right|_{s \to s-3}\right\} = te^{3t} \quad \text{and} \quad \mathscr{L}^{-1}\left\{\left.\frac{4!}{s^5}\right|_{s \to s-3}\right\} = t^4e^{3t},$$

and so (8) is $y(t) = 2e^{3t} + 11te^{3t} + \frac{1}{12}t^4e^{3t}$.



Example 4 **An Initial-Value Problem**

Solve $y'' + 4y' + 6y = 1 + e^{-t}$, $\quad y(0) = 0, \quad y'(0) = 0.$

SOLUTION $\mathscr{L}\{y''\} + 4\mathscr{L}\{y'\} + 6\mathscr{L}\{y\} = \mathscr{L}\{1\} + \mathscr{L}\{e^{-t}\}$

$$s^2Y(s) - sy(0) - y'(0) + 4[sY(s) - y(0)] + 6Y(s) = \frac{1}{s} + \frac{1}{s+1}$$

$$(s^2 + 4s + 6)Y(s) = \frac{2s+1}{s(s+1)}$$

$$Y(s) = \frac{2s+1}{s(s+1)(s^2+4s+6)}.$$

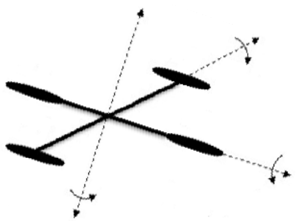
Since the quadratic term in the denominator does not factor into real linear factors, the partial fraction decomposition for $Y(s)$ is found to be

$$Y(s) = \frac{1/6}{s} + \frac{1/3}{s+1} - \frac{s/2 + 5/3}{s^2 + 4s + 6}.$$

Moreover, in preparation for taking the inverse transform, we have already manipulated the last term into the necessary form in part **(b)** of Example 2. So in view of the results in (6) and (7) we have the solution

$$y(t) = \frac{1}{6}\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2+2}\right\} - \frac{2}{3\sqrt{2}}\mathscr{L}^{-1}\left\{\frac{\sqrt{2}}{(s+2)^2+2}\right\}$$

$$= \frac{1}{6} + \frac{1}{3}e^{-t} - \frac{1}{2}e^{-2t}\cos\sqrt{2}t - \frac{\sqrt{2}}{3}e^{-2t}\sin\sqrt{2}t.$$ ❑



DEFINITION 4.3 Unit Step Function

The **unit step function** $\mathcal{U}(t-a)$ is defined to be

$$\mathcal{U}(t-a)=\begin{cases}0, & 0\le t<a\\ 1, & t\ge a.\end{cases}$$

THEOREM 4.7 Second Translation Theorem

If $F(s)=\mathscr{L}\{f(t)\}$ and $a>0$, then

$$\mathscr{L}\{f(t-a)\mathcal{U}(t-a)\}=e^{-as}F(s).$$

$$f(t-a)\mathcal{U}(t-a)=\begin{cases}0, & 0\le t<a\\ f(t-a), & t\ge a\end{cases}$$

$$\mathscr{L}^{-1}\{e^{-as}F(s)\}=f(t-a)\mathcal{U}(t-a).$$

PROOF By the additive interval property of integrals, $\int_0^\infty e^{-st} f(t-a)\mathcal{U}(t-a)\,dt$ can be written as two integrals:

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = \int_0^a e^{-st} f(t-a)\underbrace{\mathcal{U}(t-a)}_{\text{zero for } 0 \le t < a}\,dt + \int_a^\infty e^{-st} f(t-a)\underbrace{\mathcal{U}(t-a)}_{\text{one for } t \ge a}\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt.$$

Now if we let $v = t - a$, $dv = dt$ in the last integral, then

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = \int_0^\infty e^{-s(v+a)} f(v)\,dv$$

$$= e^{-as}\int_0^\infty e^{-sv} f(v)\,dv = e^{-as}\mathcal{L}\{f(t)\}. \quad \square$$

Example 6 Using Formula (15)

Evaluate (a) $\mathscr{L}^{-1}\left\{\dfrac{1}{s-4}e^{-2s}\right\}$ (b) $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+9}e^{-\pi s/2}\right\}$.

SOLUTION (a) With the identifications $a = 2$, $F(s) = 1/(s-4)$, $\mathscr{L}^{-1}\{F(s)\} = e^{4t}$, we have from (15)

$$\mathscr{L}^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\} = e^{4(t-2)}\mathscr{U}(t-2).$$

(b) With $a = \pi/2$, $F(s) = s/(s^2+9)$, $\mathscr{L}^{-1}\{F(s)\} = \cos 3t$, (15) yields

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+9}e^{-\pi s/2}\right\} = \cos 3\left(t-\frac{\pi}{2}\right)\mathscr{U}\left(t-\frac{\pi}{2}\right).$$

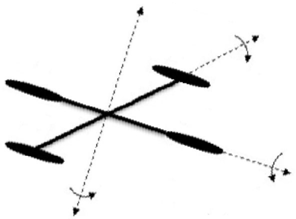
The last expression can be simplified somewhat using the addition formula for the cosine. Verify that the result is the same as $-\sin 3t\,\mathscr{U}(t-\pi/2)$. ❑

Example 7 **Second Translation Theorem—Alternative Form**

Evaluate $\mathscr{L}\{\sin t\,\mathscr{U}(t-\pi)\}$.

SOLUTION With $g(t) = \sin t$, $a = \pi$, then $g(t+\pi) = \sin(t+\pi) = -\cos t$ by the addition formula for the sine function. Hence by (16),

$$\mathscr{L}\{\sin t\,\mathscr{U}(t-\pi)\} = -e^{-\pi s}\mathscr{L}\{\cos t\} = -\frac{s}{s^2+1}e^{-\pi s}.$$



Example 8 **An Initial-Value Problem**

Solve $y' + y = f(t), \quad y(0) = 5, \quad$ where $\quad f(t) = \begin{cases} 0, & 0 \le t < \pi, \\ 3\sin t, & t \ge \pi. \end{cases}$

SOLUTION The function f can be written as $f(t) = 3\sin t\,\mathcal{U}(t-\pi)$ and so by linearity, the results of Example 7, and the usual partial fractions, we have

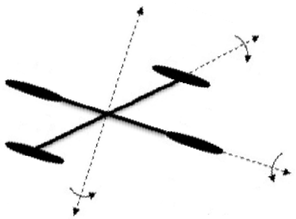
$$\mathscr{L}\{y'\} + \mathscr{L}\{y\} = 3\mathscr{L}\{\sin t\,\mathcal{U}(t-\pi)\}$$

$$sY(s) - y(0) + Y(s) = -3\,\frac{s}{s^2+1}\,e^{-\pi s}$$

$$(s+1)Y(s) = 5 - \frac{3s}{s^2+1}\,e^{-\pi s}$$

$$Y(s) = \frac{5}{s+1} - \frac{3}{2}\left[-\frac{1}{s+1}\,e^{-\pi s} + \frac{1}{s^2+1}\,e^{-\pi s} + \frac{s}{s^2+1}\,e^{-\pi s}\right]. \qquad (17)$$

Now proceeding as we did in Example 6, it follows from (15) with $a = \pi$ that the inverses of the terms in the bracket are



$$Y(s) = \frac{5}{s+1} - \frac{3}{2}\left[-\frac{1}{s+1}e^{-\pi s} + \frac{1}{s^2+1}e^{-\pi s} + \frac{s}{s^2+1}e^{-\pi s}\right]. \qquad (17)$$

Now proceeding as we did in Example 6, it follows from (15) with $a = \pi$ that the inverses of the terms in the bracket are

$$\mathscr{L}^{-1}\left\{\frac{1}{s+1}e^{-\pi s}\right\} = e^{-(t-\pi)}\mathscr{U}(t-\pi), \qquad \mathscr{L}^{-1}\left\{\frac{1}{s^2+1}e^{-\pi s}\right\} = \sin(t-\pi)\mathscr{U}(t-\pi),$$

and
$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}e^{-\pi s}\right\} = \cos(t-\pi)\mathscr{U}(t-\pi).$$

Thus the inverse of (17) is

$$y(t) = 5e^{-t} + \frac{3}{2}e^{-(t-\pi)}\mathscr{U}(t-\pi) - \frac{3}{2}\sin(t-\pi)\mathscr{U}(t-\pi) - \frac{3}{2}\cos(t-\pi)\mathscr{U}(t-\pi)$$

$$= 5e^{-t} + \frac{3}{2}\left[e^{-(t-\pi)} + \sin t + \cos t\right]\mathscr{U}(t-\pi) \qquad \leftarrow \text{trigonometric identities}$$

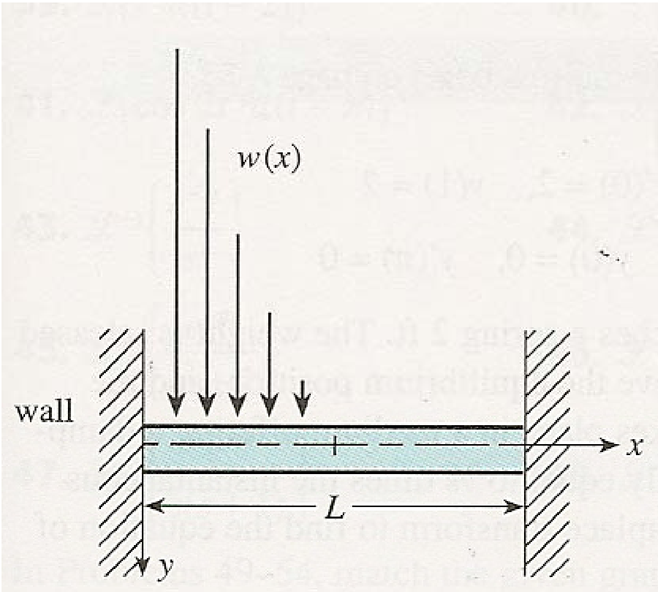
$$= \begin{cases} 5e^{-t}, & 0 \le t < \pi \\ 5e^{-t} + \frac{3}{2}e^{-(t-\pi)} + \frac{3}{2}\sin t + \frac{3}{2}\cos t, & t \ge \pi. \end{cases} \qquad (18)$$

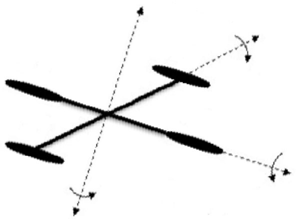
With the aid of a graphing utility we get the graph of (18), shown in Figure 4.16. ❑

Example 9 **A Boundary-Value Problem**

A beam of length L is embedded at both ends as shown in Figure 4.17. Find the deflection of the beam when the load is given by

$$w(x) = \begin{cases} w_0\left(1 - \dfrac{2}{L}x\right), & 0 < x < L/2 \\ 0, & L/2 < x < L. \end{cases}$$





SOLUTION Recall that, since the beam is embedded at both ends, the boundary conditions are $y(0) = 0$, $y'(0) = 0$, $y(L) = 0$, $y'(L) = 0$. Now by (10) we can express $w(x)$ in terms of the unit step function:

$$w(x) = w_0\left(1 - \frac{2}{L}x\right) - w_0\left(1 - \frac{2}{L}x\right)\mathcal{U}\left(x - \frac{L}{2}\right)$$

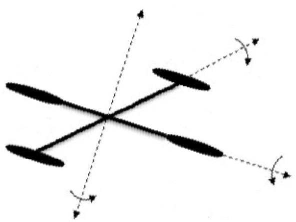
$$= \frac{2w_0}{L}\left[\frac{L}{2} - x + \left(x - \frac{L}{2}\right)\mathcal{U}\left(x - \frac{L}{2}\right)\right].$$

Transforming (9) with respect to the variable x gives

$$EI(s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)) = \frac{2w_0}{L}\left[\frac{L/2}{s} - \frac{1}{s^2} + \frac{1}{s^2}e^{-Ls/2}\right]$$

or

$$s^4Y(s) - sy''(0) - y'''(0) = \frac{2w_0}{EIL}\left[\frac{L/2}{s} - \frac{1}{s^2} + \frac{1}{s^2}e^{-Ls/2}\right].$$



If we let $c_1 = y''(0)$ and $c_2 = y'''(0)$, then

$$Y(s) = \frac{c_1}{s^3} + \frac{c_2}{s^4} + \frac{2w_0}{EIL}\left[\frac{L/2}{s^5} - \frac{1}{s^6} + \frac{1}{s^6}e^{-Ls/2}\right],$$

and consequently

$$y(x) = \frac{c_1}{2!}\mathscr{L}^{-1}\left\{\frac{2!}{s^3}\right\} + \frac{c_2}{3!}\mathscr{L}^{-1}\left\{\frac{3!}{s^4}\right\}$$

$$+ \frac{2w_0}{EIL}\left[\frac{L/2}{4!}\mathscr{L}^{-1}\left\{\frac{4!}{s^5}\right\} - \frac{1}{5!}\mathscr{L}^{-1}\left\{\frac{5!}{s^6}\right\} + \frac{1}{5!}\mathscr{L}^{-1}\left\{\frac{5!}{s^6}e^{-Ls/2}\right\}\right]$$

$$= \frac{c_1}{2}x^2 + \frac{c_2}{6}x^3 + \frac{w_0}{60EIL}\left[\frac{5L}{2}x^4 - x^5 + \left(x - \frac{L}{2}\right)^5 \mathscr{U}\left(x - \frac{L}{2}\right)\right].$$

Applying the conditions $y(L) = 0$ and $y'(L) = 0$ to the last result yields a system of equations for c_1 and c_2:

$$c_1\frac{L^2}{2} + c_2\frac{L^3}{6} + \frac{49w_0L^4}{1920EI} = 0$$

$$c_1L + c_2\frac{L^2}{2} + \frac{85w_0L^3}{960EI} = 0.$$

Solving, we find $c_1 = 23w_0L^2/960EI$ and $c_2 = -9w_0L/40EI$. Thus the deflection is given by

$$y(x) = \frac{23w_0L^2}{1920EI}x^2 - \frac{3w_0L}{80EI}x^3 + \frac{w_0}{60EIL}\left[\frac{5L}{2}x^4 - x^5 + \left(x - \frac{L}{2}\right)^5 \mathscr{U}\left(x - \frac{L}{2}\right)\right]. \quad \square$$