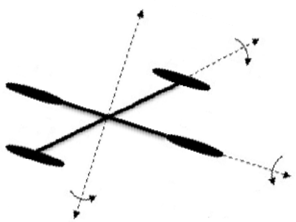
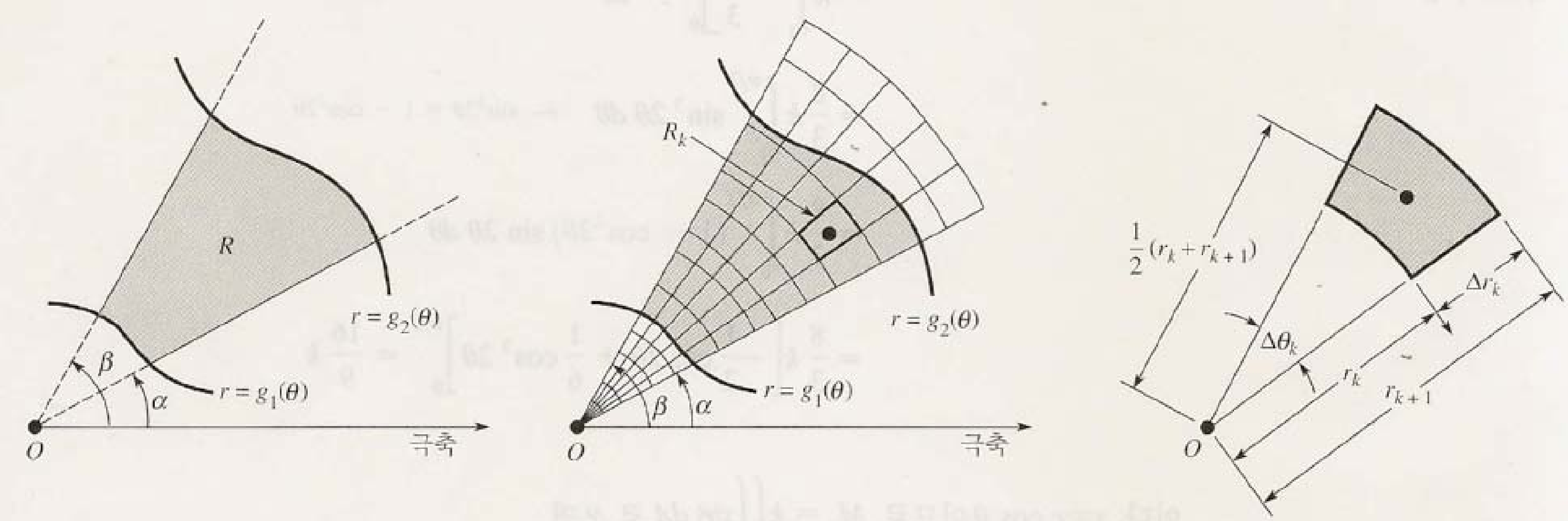
9장 벡터의 미적분
(9)

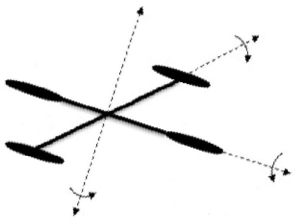


- 극좌표계에서의 이중적분 - 극사각형



(a) 영역 R이 극방정식의 그래프와 사선으로 제한됨
(b) R_k 부분영역
(c) R_k의 확대

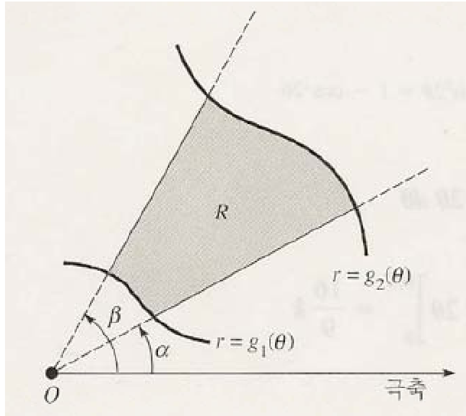
$$\Delta A_k = \frac{1}{2}(r_{k+1}^2 - r_k^2)\Delta\theta_k = \frac{1}{2}(r_{k+1} + r_k)(r_{k+1} - r_k)\Delta\theta_k = r_k^* \Delta r_k \Delta\theta_k$$



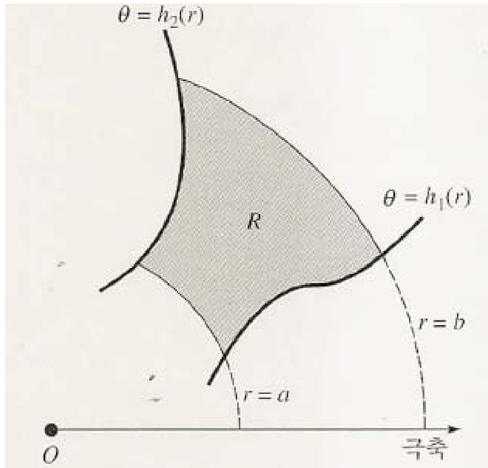
$$\Delta A_k = \tfrac{1}{2}(r_{k+1}^2 - r_k^2)\Delta\theta_k = \tfrac{1}{2}(r_{k+1} + r_k)(r_{k+1} - r_k)\Delta\theta_k = r_k^*\Delta r_k\Delta\theta_k$$

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(r_k^*, \theta_k^*) r_k^* \, \Delta r_k \, \Delta\theta_k = \iint_R f(r, \theta) \, dA$$

$$\iint_R f(r, \theta) \, dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r, \theta) \, r \, dr \, d\theta$$

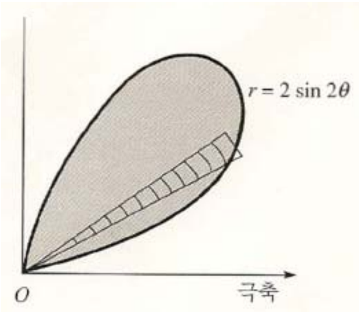


$$\iint_R f(r, \theta) \, dA = \int_a^b \int_{h_1(\theta)}^{h_2(\theta)} f(r, \theta) \, r \, d\theta \, dr$$



예제 1 질량중심

점 P의 밀도가 극점으로부터의 거리에 비례할 때 제1사분면에서 장미 한 잎 $r=2\sin 2\theta$로 둘러싸인 영역에 대응하는 얇은 판의 질량중심을 구하라.



$d(O, P)=|r|$

$\rho(\mathbf{r}, \theta)=k|r|$($k$는 비례상수)

$$m = \iint_R k|r|\,dA$$

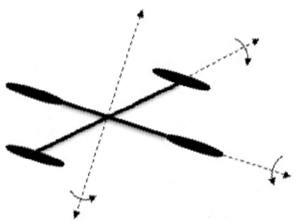
$$= k\int_0^{\pi/2}\int_0^{2\sin 2\theta} (r)r\,dr\,d\theta$$

$$= k\int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^{2\sin 2\theta} d\theta$$

$$= \frac{8}{3}k\int_0^{\pi/2} \sin^3 2\theta\,d\theta \quad \leftarrow \sin^2 2\theta = 1 - \cos^2 2\theta$$

$$= \frac{8}{3}k\int_0^{\pi/2} (1-\cos^2 2\theta)\sin 2\theta\,d\theta$$

$$= \frac{8}{3}k\left[-\frac{1}{2}\cos 2\theta + \frac{1}{6}\cos^3 2\theta\right]_0^{\pi/2} = \frac{16}{9}k$$



이다. $x=r\cos\theta$ 이므로 $M_y = k\iint_R x|r|\,dA$ 로 쓰면

$$M_y = \int_0^{\pi/2}\int_0^{2\sin 2\theta} r^3\cos\theta\,dr\,d\theta$$

$$= k\int_0^{\pi/2}\left[\frac{r^4}{4}\cos\theta\right]_0^{2\sin 2\theta} d\theta$$

$$= 4k\int_0^{\pi/2}\sin^4 2\theta\cos\theta\,d\theta \quad \leftarrow \text{배각공식}$$

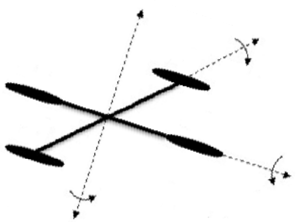
$$= 4k\int_0^{\pi/2} 16\sin^4\theta\cos^4\theta\cos\theta\,d\theta$$

$$= 64k\int_0^{\pi/2}\sin^4\theta\cos^5\theta\,d\theta$$

$$= 64k\int_0^{\pi/2}\sin^4\theta(1-\sin^2\theta)^2\cos\theta\,d\theta$$

$$= 64k\int_0^{\pi/2}(\sin^4\theta - 2\sin^6\theta + \sin^8\theta)\cos\theta\,d\theta$$

$$= 64k\left[\frac{1}{5}\sin^5\theta - \frac{2}{7}\sin^7\theta + \frac{1}{9}\sin^9\theta\right]_0^{\pi/2} = \frac{512}{315}k$$



이다. 유사한 방법으로 $y=r\sin\theta$ 이므로

$$M_x = k\int_0^{\pi/2}\int_0^{2\sin 2\theta} r^2 \sin\theta \, dr\, d\theta = \frac{512}{315}k$$

를 구한다.* 따라서 질량중심의 직교좌표는

$$\bar{x} = \bar{y} = \frac{512k/315}{16k/9} = \frac{32}{35}$$

이다. ❑

- 극좌표계에서의 이중적분 – 직교좌표를 극좌표로

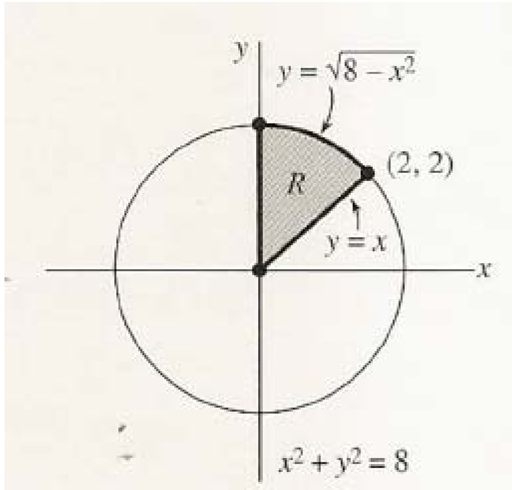
$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

예제 2 이중적분의 극좌표계 변환

극좌표계를 사용하여

$$\int_0^2 \int_x^{\sqrt{8-x^2}} \frac{1}{5 + x^2 + y^2}\, dy\, dx$$

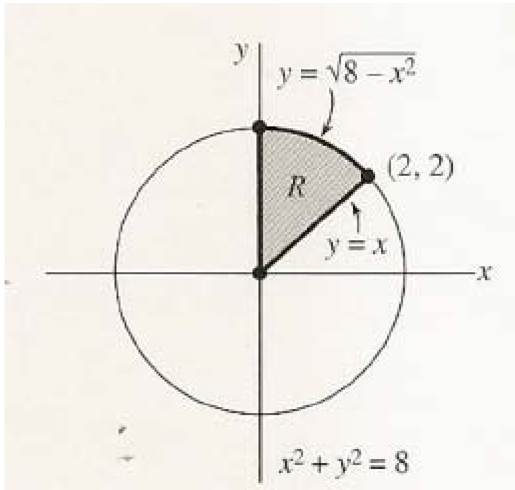
를 계산하라.



풀이 $x \le y \le \sqrt{8-x^2}$, $0 \le x \le 2$로부터 적분 영역 R은 그림 9.84와 같다. $x^2+y^2=r^2$이므로 원 $x^2+y^2=8$에 대한 극방정식은 $r=\sqrt{8}$이다. 따라서 극좌표계에서 영역 R은 $0 \le r \le \sqrt{8}$, $\pi/4 \le \theta \le \pi/2$이고 $1/(5+x^2+y^2)=1/(5+r^2)$이므로

$$\int_0^2 \int_x^{\sqrt{8-x^2}} \frac{1}{5+x^2+y^2}\,dy\,dx = \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{8}} \frac{1}{5+r^2}\,r\,dr\,d\theta$$

$$= \frac{1}{2} \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{8}} \frac{2r\,dr}{5+r^2}\,d\theta$$



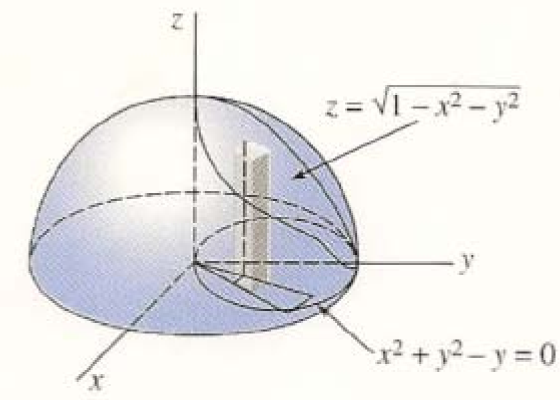
$$= \frac{1}{2}\int_{\pi/4}^{\pi/2} \ln(5 + r^2)\Big]_0^{\sqrt{8}}\, d\theta$$

$$= \frac{1}{2}(\ln 13 - \ln 5)\int_{\pi/4}^{\pi/2} d\theta$$

$$= \frac{1}{2}(\ln 13 - \ln 5)\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{8}\ln\frac{13}{5}$$

예제 3 부피

반구 $z=\sqrt{1-x^2-y^2}$의 아래, 원 $x^2+y^2-y=0$의 위로 제한되는 입체의 부피를 구하라.



풀이 그림 9.85에서 $V=\iint_R \sqrt{1-x^2-y^2}\,dA$임을 알 수 있다. 극좌표계에서 반구와 원의 방정식은 각각 $z=\sqrt{1-r^2}$과 $r=\sin\theta$가 된다. 대칭성을 이용하면

$$V=\iint_R \sqrt{1-r^2}\,dA = 2\int_0^{\pi/2}\int_0^{\sin\theta}(1-r^2)^{1/2}r\,dr\,d\theta$$

$$=2\int_0^{\pi/2}\left[-\frac{1}{3}(1-r^2)^{3/2}\right]_0^{\sin\theta}d\theta$$

$$=\frac{2}{3}\int_0^{\pi/2}[1-(1-\sin^2\theta)^{3/2}]\,d\theta$$

$$=\frac{2}{3}\int_0^{\pi/2}[1-(\cos^2\theta)^{3/2}]\,d\theta$$

$$=\frac{2}{3}\int_0^{\pi/2}[1-\cos^3\theta]\,d\theta$$

$$=\frac{2}{3}\int_0^{\pi/2}[1-(1-\sin^2\theta)\cos\theta]\,d\theta$$

$$=\frac{2}{3}\left[\theta-\sin\theta+\frac{1}{3}\sin^3\theta\right]_0^{\pi/2}=\frac{\pi}{3}-\frac{4}{9}\approx 0.60$$

이다. ❑